A more quantitative proof of Rouché's Thm.

Given f, g holo on domain D,
γ curve inside D with $\text{inter}(\gamma)$ inside D

If $|f(z)| > |g(z)| \quad \forall z \in \gamma$, then

$$Z_{f+g} = Z_f$$

(ie f & $f+g$ have the same # of zeros inside γ, counting multiplicities)

Proof

$$Z_{f+g} - Z_f \overset{\text{Argument Principle}}{=} \frac{1}{2\pi i} \int_\gamma \frac{(f+g)'}{f+g} - \frac{f'}{f}$$

$$= \frac{1}{2\pi i} \int_\gamma \big(\log(f+g) - \log f \big)'$$

any branch of log.

$$= \frac{1}{2\pi i} \int_\gamma \left(\log \left(\frac{f+g}{f} \right) \right)'$$

$$= \frac{1}{2\pi i} \int_\gamma \left(\log \left(1 + \frac{g(z)}{f(z)} \right) \right)' dz$$

$\left|\frac{g}{f}\right| < 1$

∴ $1+\frac{g(z)}{f(z)}$ ~~is~~ in open right half-plane

(Actually in open disk centered at 1 of radius 1)

$$\therefore\ Z_{f+g} - Z_f = \frac{1}{2\pi i}\int_\gamma \underbrace{\left(\log\left(1+\frac{g(z)}{f(z)}\right)\right)'}_{\text{holomorphic}} dz$$

Choose principal branch

$$= \boxed{0}.$$

<u>Application</u>. How many zeros of

$$p(z) = z^{11} - 25z^2 + 15z - 5$$

are in the unit disk?

Let γ = unit circle, CCW. Observe that if $|z|=1$,

$$|\underbrace{-25z^2}_{f}| = 25 > 21 = |z|^{11} + 15|z| + 5 \ge |\underbrace{z^{11}+15z-5}_{g}|$$

$\therefore$ $(f+g)(z) = z^{11} - 25z^2 + 15z - 5$
has the same # of zeros in the unit disk (counting multiplicities) as $f(z) = -25z^2$ ← one zero of mult. 2.

$\Rightarrow$ $\boxed{(f+g)(z) = p(z) \text{ has two zeros inside the unit disk!}}$

Gamma function:

$\Gamma(z) = (z-1)!$ if z is a positive integer.

$\Gamma(3) = 2! = 2$

$\Gamma(7) = 6! = 720$

$\Gamma(1) = 0! = 1$

$n\Gamma(n) = n \cdot (n-1)! = n! = \Gamma(n+1)$ if n is a positive integer.

Another way to write $\Gamma(n)$.

$$\Gamma(n) = (n-1)! = \int_0^\infty t^{n-1} e^{-t}\,dt$$ (can check that)

Let $z \in \mathbb{C}$, $\operatorname{Re} z > 0$

Then $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\,dt$

actually converges (on $\{z : \mathrm{Re}\, z > 0\}$)

$$t^{z-1} = t^{x+iy-1} = t^{iy} \cdot t^{(x-1)}$$

$t^{iy} = e^{(\log t) iy}$ ← norm 1.

$t^{(x-1)}$ ← abs value

$t^{z-1} = e^{(\log t)(z-1)}$ is holomorphic.

& in fact the whole integral is holomorphic

$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\,dt$ is

a holomorphic fcn of z for $\mathrm{Re}(z) > 1$.

This is a holomorphic generalization of

$(n-1)! = \Gamma(n)$.

Using this equation, you can prove

(~~variable substitution~~ integrate by parts)

$$z\Gamma(z) = \Gamma(z+1).$$

$$\Rightarrow \Gamma(z) = \frac{\Gamma(z+1)}{z}$$

← defined

This is an analytic continuation of $\Gamma(z)$ to $\mathrm{Re}(z) > -1$.

(with pole at $z = 0$)

This means $\Gamma(z+1) = \dfrac{\Gamma(z+2)}{z+1}$ (plugged $(z+1)$ into z in last eqn)

$$\Rightarrow \Gamma(z) = \frac{\Gamma(z+1)}{z} = \frac{\Gamma(z+2)}{z(z+1)}$$

Keep doing this:

$$\Gamma(z) = \frac{\overbrace{\Gamma(z+k)}^{\text{holom.}}}{\underbrace{z(z+1)(z+2)\cdots(z+k-1)}_{\text{holomorphic } \{z \mid Re(z) > -k\} \setminus \{0,-1,-2,\dots,-(k-1)\}}} \qquad k \in \mathbb{N}$$

$z - (-(k-1))$
$z+k-1$

By using this we continue $\Gamma(z)$ to be holomorphic on $\mathbb{C} \setminus \{0,-1,-2,\dots\}$ with simple poles at $z=0,-1,-2,\dots$

---

like $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$

Another Amazing function

Riemann zeta function

$$\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$$ ← converges absolutely if $Re(z) > 1$

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3} = ? \ldots$$

If $Re(z) > 1$ $z = x + iy$, $x > 1$, $y \in \mathbb{R}$

$$\zeta(z) = \sum_{n=1}^{\infty} n^{-z} = \sum_{n=1}^{\infty} n^{-x} \underbrace{n^{-iy}}_{\left(e^{-\log(n) iy}\right)}$$

$x > 1$

$$|n^{-z}| = n^{-x}$$

$$\sum_{n=1}^{\infty} |n^{-z}| = \sum_{n=1}^{\infty} \frac{1}{n^x}$$

converges abs. if $x > 1$.

In fact $\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$ is holomorphic

on $\underline{\{z : \mathrm{Re}(z) > 1\}}$.

How can we analytically continue $\zeta(z)$?

Interesting fact:

$$\underline{n^{-z} = \frac{1}{\Gamma(z)} \int_0^{\infty} t^{z-1} e^{-nt} \, dt}$$

Proof: $\int_0^{\infty} t^{z-1} e^{-nt} \, dt \leftarrow$ let $u = nt$

$du = n\,dt$

$\frac{1}{n} du = dt$

$$\int_0^{\infty} \left(\frac{u}{n}\right)^{z-1} e^{-u} \frac{1}{n} \, dt$$

$$= \frac{1}{n^z} \underbrace{\int_0^{\infty} u^{z-1} e^{-u} \, dt}_{\Gamma(z)} . \quad \boxtimes$$

$\therefore$

$$\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$$

$$= \sum_{n=1}^{\infty} \frac{1}{\Gamma(z)} \int_0^{\infty} t^{z-1} e^{-nt} \, dt$$

$$= \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1} \Bigg(\underbrace{\sum_{n=1}^{\infty} \underbrace{e^{-nt}}_{(e^{-t})^n}}_{\substack{a=e^{-t} \\ r=e^{-t}}} \Bigg) dt$$

(everything converges absolutely, no prob.)

$$= \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1} \overbrace{\frac{e^t (e^{-t})}{e^t (1-e^{-t})}}^{\frac{a}{1-r}} dt$$

$$\zeta(z) = \frac{1}{\Gamma(z)} \int_0^\infty \frac{t^{z-1}}{e^t - 1} dt$$

Works for $Re(z) > 1$

$$= \frac{1}{\Gamma(z)} \int_0^\varepsilon \frac{t^{z-1}}{e^t-1} dt$$

for $\varepsilon > 0$

$$+ \underbrace{\frac{1}{\Gamma(z)} \int_\varepsilon^\infty \frac{t^{z-1}}{e^t-1} dt}_{\text{holomorphic in } z}$$

$\forall z \in \mathbb{C}$.

Consider $\int_0^\varepsilon \frac{t^{z-1}}{e^t-1} dt = \int_0^\varepsilon t^{z-1} \left(\frac{1}{t} (a_0 + a_1 t + \ldots) \right) dt$

$\frac{1}{e^t-1}$ meromorph (simple pole at $t=0$)

$$= \int_0^{\varepsilon} a_0 t^{z-2} + a_1 t^{z-1} + a_2 t^{z} + a_3 t^{z+1} \cdots \, dt$$

$$= \frac{a_0 \varepsilon^{z-1}}{z-1} + \frac{a_1 \varepsilon^{z}}{z} + \frac{a_2 \varepsilon^{z+1}}{z+1} + \cdots$$

$\therefore$ $\varepsilon = 1$

$$\zeta(z) = \frac{1}{\Gamma(z)} \left[\frac{a_0^{=1}}{z-1} + \frac{a_1}{z} + \frac{a_2}{z+1} + \cdots \right]$$

$$+ \underbrace{\frac{1}{\Gamma(z)} \int_1^{\infty} \frac{t^{z-1}}{e^t - 1} \, dt}_{\text{holom. on } \mathbb{C}.}$$

$\Gamma(z)$ is never zero

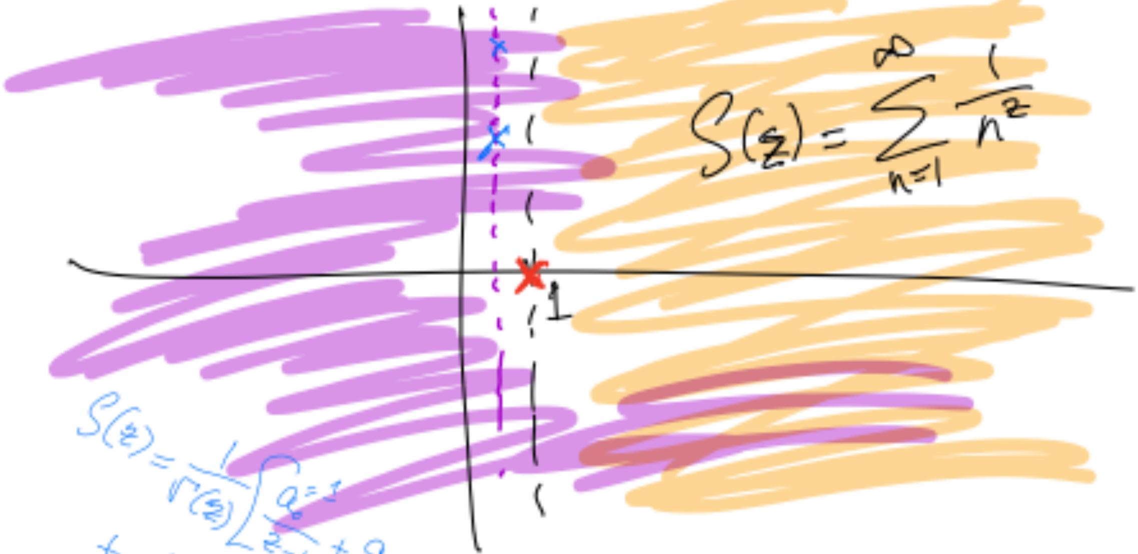
$\Gamma(z)$ has simple poles at $z = 0, -1, -2, \cdots$

$\Rightarrow \frac{1}{\Gamma(z)}$ is holomorphic

$\frac{1}{\Gamma(z)}$ has simple zeros at $z = 0, -1, -2, \cdots$

$\therefore$ This expression for $\zeta(z)$ is holomorphic on $\mathbb{C} \setminus \{1\}$ $\to$ simple pole at $z = 1$.

(Note: $\operatorname{Res}(\zeta(z), 1) = 1$.)



$$\zeta(z) = \frac{1}{\Gamma(z)}\left[\frac{a_0}{z-1} + \frac{a_1}{z} + \frac{a_2}{z+1} + \dots\right] + \frac{1}{\Gamma(z)}\int_1^\infty \frac{t^{z-1}}{e^t - 1}\,dt \qquad a_0 = 1$$

Riemann: make the connection between prime numbers & $\zeta(z)$.

Famous conjecture: Except for some zeros of $\zeta(z)$ on the negative real axis — all the zeros of $\zeta(z)$ are on the "critical line" $\operatorname{Re}(z) = \frac{1}{2}$

(Riemann hypothesis)

Connection with prime #s.

$$\sum \frac{1}{n^z}$$

$$\left(1+\frac{1}{p_1^z}+\frac{1}{p_1^{2z}}+\cdots\right)\left(1+\frac{1}{p_2^z}+\frac{1}{p_2^{2z}}+\cdots\right)\cdots\left(\qquad\right)$$

primes $p_1, p_2, p_3, \dots$

$$= \prod_{\substack{\text{primes}\\ p}}\left(\underbrace{1+\frac{1}{p^z}+\frac{1}{p^{2z}}+\cdots}_{\text{geometric seri.}}\right)$$

$a=1$

$r=\frac{1}{p^z}$ $\quad \frac{a}{1-r}=\frac{1}{1-p^{-z}}$

$$\boxed{\zeta(z)=\prod_{\substack{\text{primes}\\ p}}\left(\frac{1}{1-p^{-z}}\right)}$$

$$=\sum_{n=1}^{\infty} n^{-z}$$